The FFT

→ DFT can be used to perform linear filtering.
Set of algorithm known as FFT.
→ These algorithms popularized by Cooley & Tukey is based on decomposing & breaking the transform into smaller transform & combining them to give the total transform.
→ FFT reduces the computation time and improves the performance by a factor 100 or more over direct evaluation of the DFT.

Direct Evaluation of DFT

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}} \qquad K = 0 \cdot 1 \cdots N-1$$

$W_N = e^{-j2\pi/N}$ → twiddle factor.

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{nK} \qquad K = 0, 1, \cdots N-1.$$

* Symmetry property: $W_N^{K+N/2} = -W_N^K$

* Periodicity property: $W_N^{K+N} = W_N^K$.

Fast fourier transform

→ It is based on fundamental principle of decomposing the computation of dft of sequence of length 'N' into successively smaller DFT.

→ FFT are two types.

→ Decimation-in-time

→ Decimation-in-frequency.

Decimation-in-time algorithm (DIT)

It is known as Radix-2 DIT FFT algorithm. Which means the no. of output points 'N' can be expressed as power of '2' i.e $N = 2^M$, where M is an integer.

→ $x(n)$ is an N-point sequence.

even sequence

$x_e(n) = x(2n)$

$(\because n = 0, 1, \ldots N/2 - 1)$

odd sequence

$x_o(n) = x(2n+1)$

$n = 0, 1, \ldots N/2 - 1$.

→ The N-point DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \quad k = 0, 1, 2 \ldots N-1.$$

$$X(K) = \underbrace{\sum_{n=0}^{N-1} x(n) W_N^{nk}}_{(\text{even})} + \underbrace{\sum_{n=0}^{N-1} x(n) W_N^{nk}}_{(\text{odd})}$$

$$= \sum_{n=0}^{N/2-1} x(2n) W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \underbrace{\sum_{n=0}^{N/2-1} x(2n) W_N^{2nk}}_{\text{even}} + W_N^{k} \underbrace{\sum_{n=0}^{N/2-1} x(2n+1) W_N^{2nk}}_{\text{odd}}$$

$$\Rightarrow X(K) = \sum_{n=0}^{N/2-1} x_e(n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{N/2-1} x_o(n) W_N^{2nk}$$

$$\left[\begin{array}{l} \because \; W_N^2 = \left(e^{-j2\pi/N}\right)^2 = \left(e^{-j2\pi/N/2}\right) = W_{N/2} \\ \text{i.e. } W_N^2 = W_{N/2} \end{array} \right]$$

So, $X(K) = \underbrace{\sum_{n=0}^{N/2-1} x_e(n) W_{N/2}^{nk}}_{N/2 \text{ Point DFT of even indexed Sequence}} + W_N^{k} \underbrace{\sum x_o(n) W_{N/2}^{nk}}_{N/2\text{-Point DFT odd indexed Sequence.}}$

$X(k) = X_e(k) + X_o(k) \cdot W_N^k$

The 8-point DFT Using Radix-2 FFT

→ The computation of 8-pt DFT using Radix-2, involves 3-stages of computation.

$$\left[\therefore N = 8 = 2^{3} = r^{m} \quad \text{So, } r = 2 \ \& \ m = 3\right]$$

* The 8-pt sequence is decimated to 2-pt sequences. For each 2-pt sequence, the 2-pt DFT is computed.
→ from the result of 2-pt DFT, the 4-pt DFT can be computed. Again from the result of 4-pt DFT, the 8-pt DFT can be computed.

* The 8-samples should be decimated into sequences of 2-samples. Before decimation they ~~should be~~ are arranged in bit reversed order as shown –

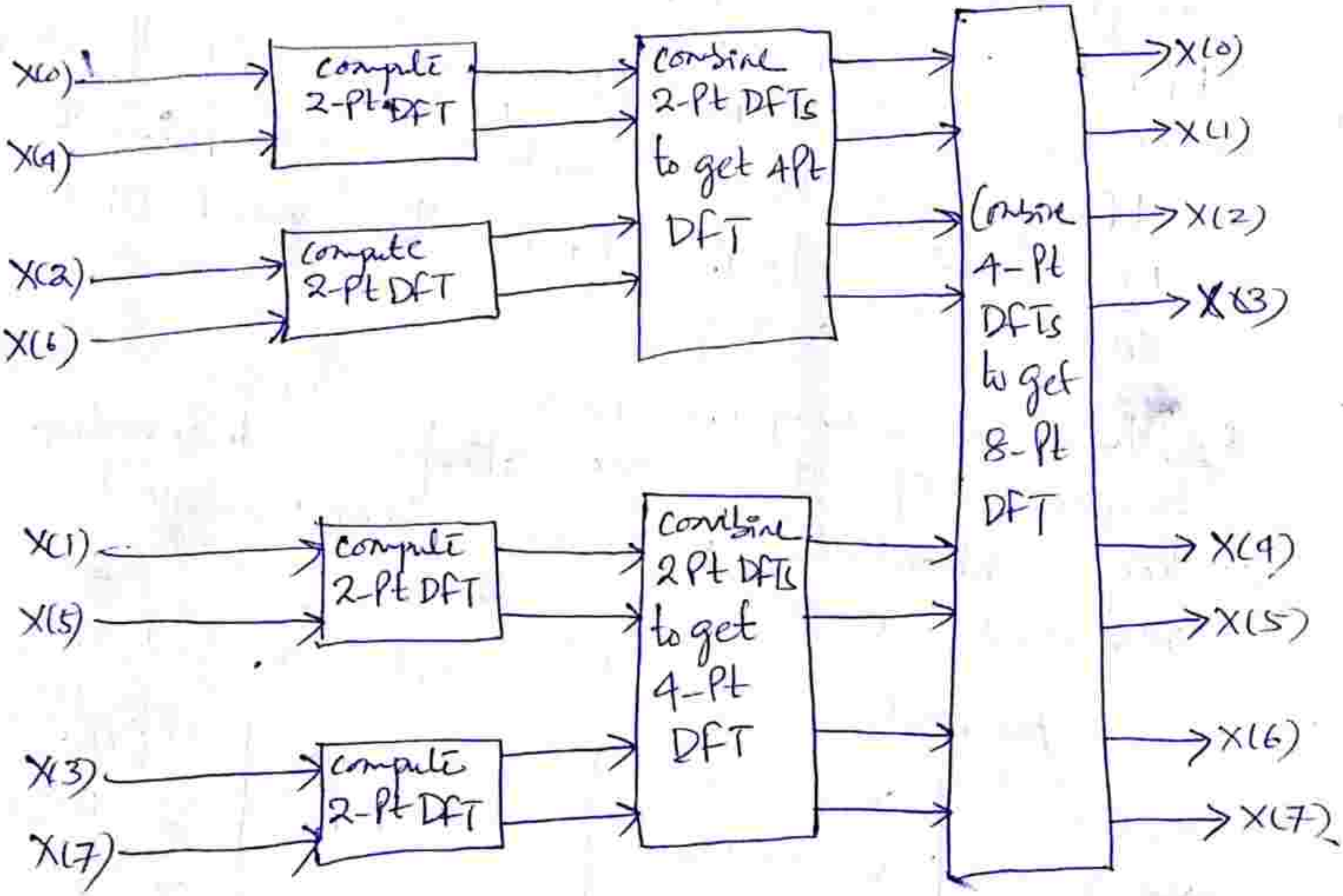
Normal order	
x(0)	x(000)
x(1)	x(001)
x(2)	x(010)
x(3)	x(011)
x(4)	x(100)
x(5)	x(101)
x(6)	x(110)
x(7)	x(111)

Bit reversed order	
x(0)	x(000)
x(4)	x(100)
x(2)	x(010)
x(6)	x(110)
x(1)	x(001)
x(5)	x(101)
x(3)	x(011)
x(7).	x(111)

→ The x(n) in bit ~~sequence~~ reversed order is decimated into 4-numbers of 2-point seque[nces] as shown below:

(i) x(0) & x(4) (ii) x(2) & x(6)
(iii) x(1) & x(5) (iv) x(3) & x(7)

Three Stages of computation of an 8-Pt DFT



Flow graph for 8-pt DIT Radix-2 FFT

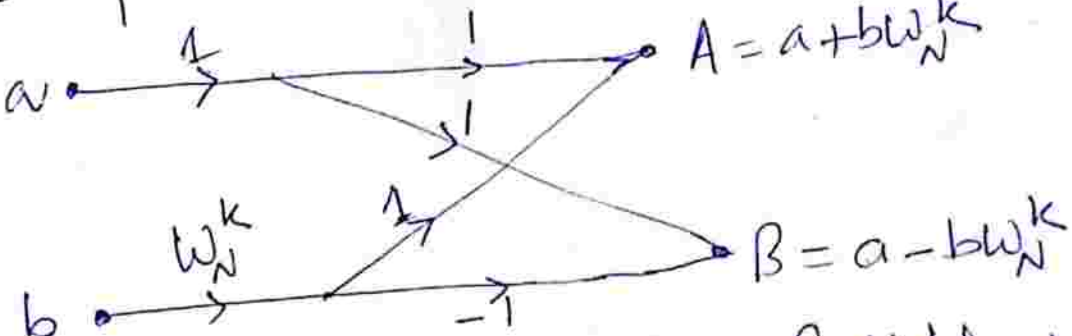
* The basic computation performed at every stage. The main conclusions are —

(i) In each computation two complex numbers 'a' & 'b' are considered.

(ii) The complex number 'b' is multiplied by a phase factor W_N^k.

(iii) The product bW_N^k is added to the complex number 'a' to form new complex number 'A'.

(iv) The product bW_N^k is subtracted from complex number 'a' to form new complex number 'B'.



Signal flow diagram or Butterfly diagram

Butterfly diagram for 8-pt DFT

The sequence is arranged in bit reversed order & then decimated into the sample sequence as

$x(0)$	$x(2)$	$x(1)$	$x(3)$
$x(4)$	$x(6)$	$x(5)$	$x(7)$.

$$W_N^k = e^{\frac{-j2\pi k}{N}}$$ twiddle factor W_N^k

$$W_8^1 = e^{\frac{-j2\pi\times 1}{8}} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4$$

$$W_8^2 = e^{\frac{-j2\pi\times 2}{8}} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$$

$$W_8^3 = e^{\frac{-j2\pi\times 3}{8}} = e^{\frac{-j3\pi}{4}} = -\frac{1}{\sqrt{2}} - j\,1/\sqrt{2}$$

$$W_4^0 = 1$$

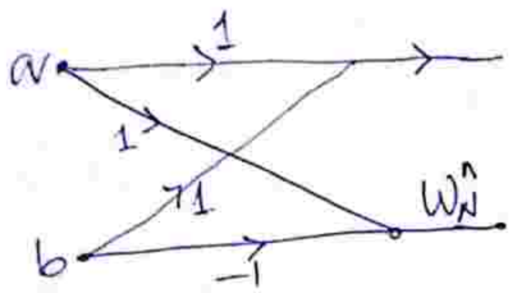
DIF Radix-2 FFT

Butterfly computation involves the following operations

→ In each computation two complex number 'a' and 'b' are considered.

→ The sum of the two complex number is computed which forms a new complex number 'A'.

→ Subtract the complex number 'b' from 'a' to get the term (a-b). The difference term (a-b) is multiplied with phase factor W_N^n to form a new complex number 'B'.



$A = a + b$.

$B = (a-b) W_N^n$.

Comparision of DIT & DIF

→ In DIT, the input is bit reversed while the output is in natural order. for DIF, the reverse is true i.e input is normal order, while the o/p bit is reverse order. Both DIT & DIF can go from normal to shuffled data or vice versa.

→ Considering the butterfly diagram, in DIF, the complex multiplication takes place after the add-substract operation.

Similarities —

→ Both algorithms require same number of operation to compute DFT.

→ Both algorithms require bit-reversal at some place during computation.

Problems Ex: An 8-pt sequence is given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$. compute 8-pt DFT of $x(n)$ by radix-2 DIF FFT.

Soln :- for 8-pt DFT by radix-2 FFT require 3-stages of computation.

→ The given sequences is the input to first stage.

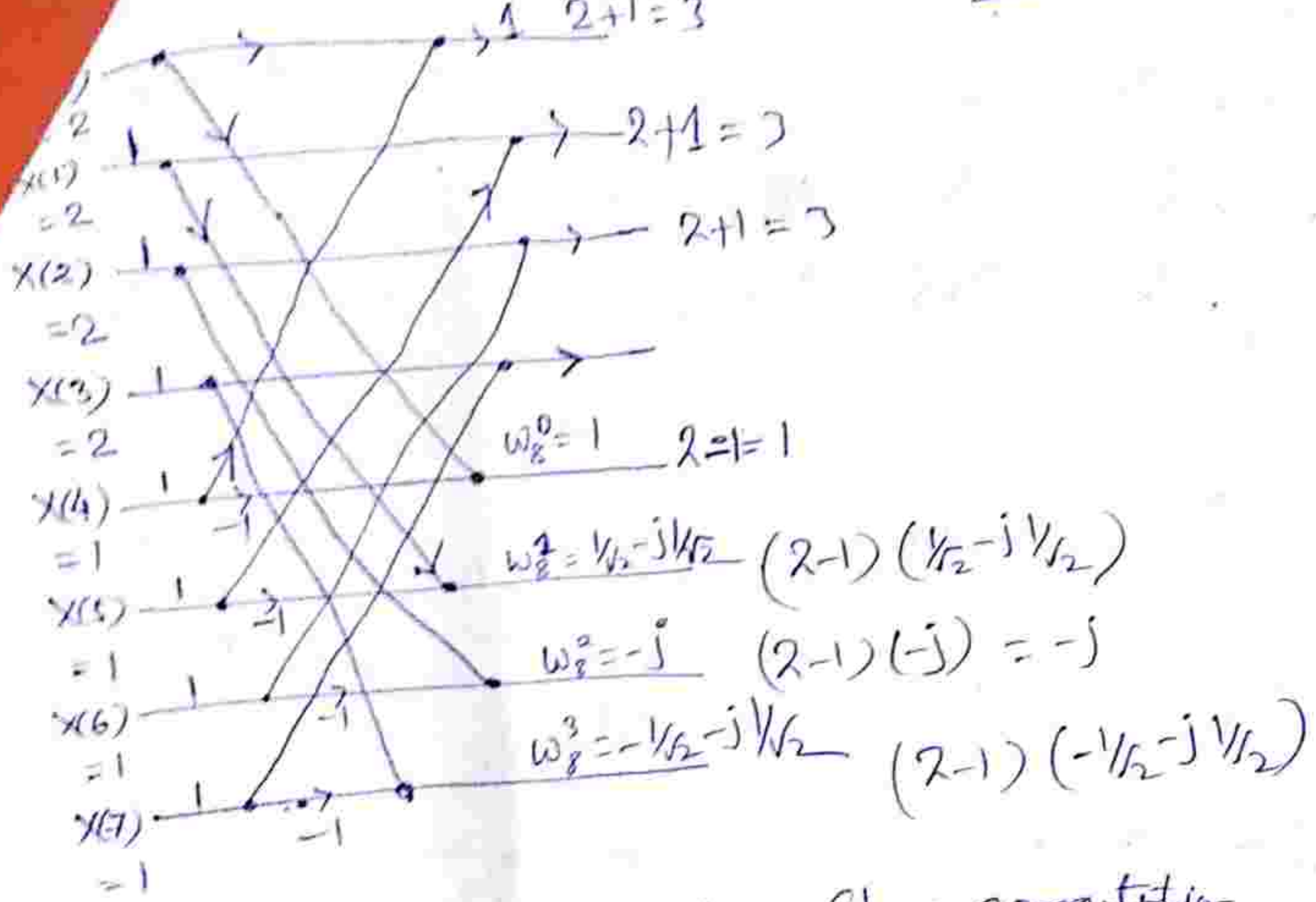
→ for other stages of computation, the o/p of the previous stage will be the input for the current stage.

* first stage computation —

Input sequence $= \{2, 2, 2, 2, 1, 1, 1, 1\}$

Phase-factor involved in the first stage computation are W_8^0, W_8^1, W_8^2 & W_8^3

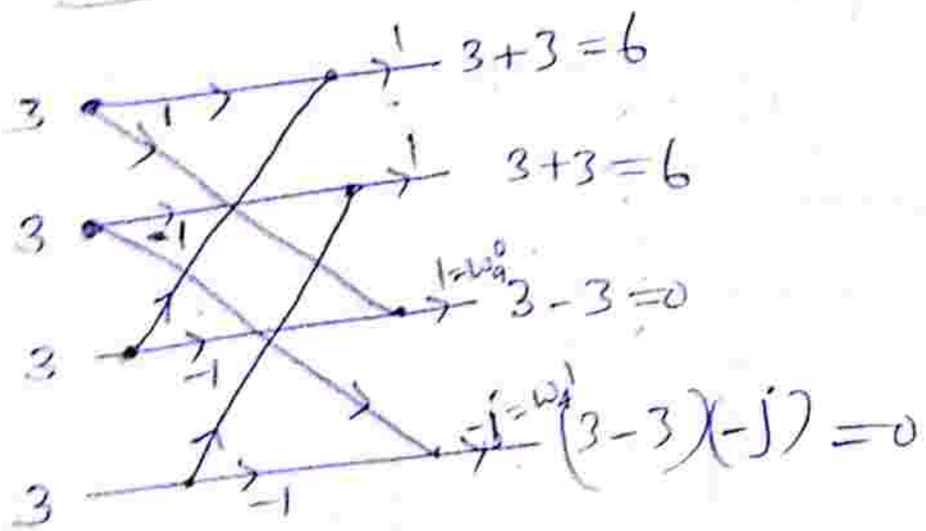
$W_8^0 = 1$ $\quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$ $\quad W_8^2 = -j$ $\quad W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$.

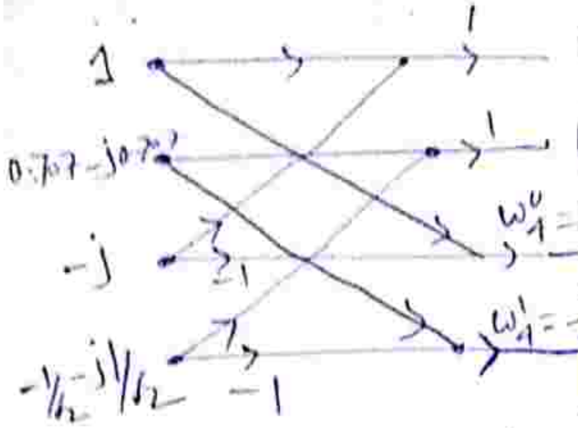


The o/p sequence of 1st stage computation

$= \{3, 3, 3, 3, \cancel{0.707}, 1, 1/\sqrt{2}-j1/\sqrt{2}, -j$
$-1/\sqrt{2}-j1/\sqrt{2}\}$

Second Stage computation.





$1+(-j) = 1-j$

$(0.707-j0.707)+(-0.707-j0.707) = -j1.414$

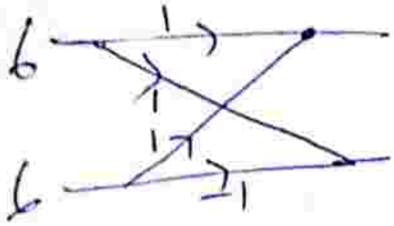
$\{1-(-j)\}1 = 1+j$

$\{(0.707-j0.707)-(-0.707-j0.707)\}(-j)$

$= -j1.414$

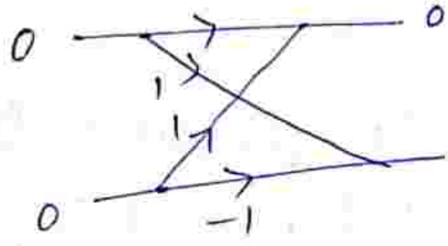
O/P Sequence = $\{6, 6, 0, 0, 1-j, -j1.414, 1+j, -j1.414\}$.

Third Stage computation



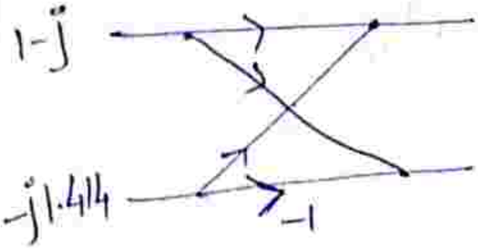
$6+6=12$
$x(0)$

$6-6=0$
$x(4)$



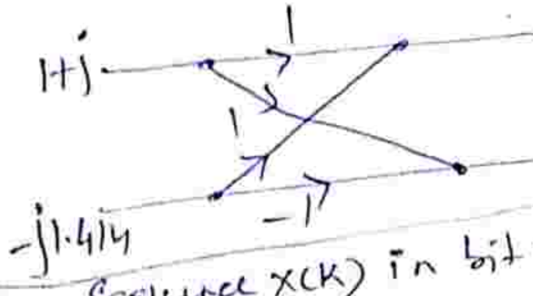
$0+0=0$
$=x(2)$

$0-0=0$
$x(6)$.



$(1-j)+(-j1.414)$
$= 1-j2.414 = x(1)$

$(1-j)-(-j1.414) = 1+j0.414 = x(5)$



$(1+j)+(-j1.414) = 1-j0.414 = x(3)$

$(1+j)-(-j1.414) = 1+j2.414 = x(7)$

Sequence x(K) in bit reversed

$x(0)=12$
$x(4)=0$
$x(2)=0$
$x(6)=0$
$x(1)=1-j2.414$
$x(5)=1+j0.414$
$x(3)=1-j0.414$
$x(7)=1+j2.414$

Sequence x(K) in Normal order

$x(0)=12$
$x(1)=1-j2.414$
$x(2)=0$
$x(3)=1-j0.414$
$x(4)=0$
$x(5)=1+j0.414$
$x(6)=0$
$x(7)=1+j2.414$.

C. Babu
Durgai
page-231

Scanned by CamScanner

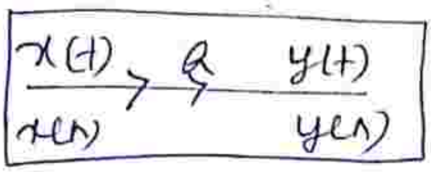
Structure Realisation of IIR Syst. Recursive Structu.

→ The block diagram consists of an interconnection of three elementary operations on the signals which are

1. Scalar Multiplication

$y(t) = a\,x(t) \Rightarrow$ CTS

$y(n) = a\,x(n) \Rightarrow$ DTS



where a is constant

2) Addition

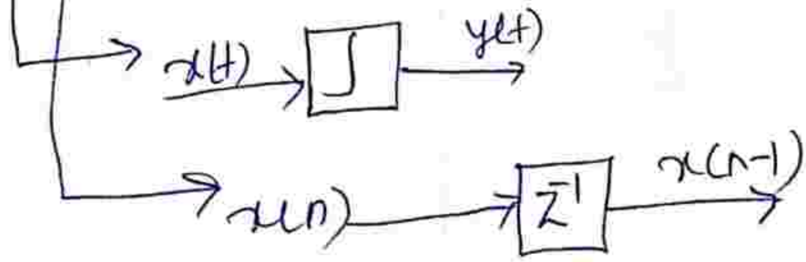
$y(t) = x(t) + w(t) \Rightarrow$ CTS

$y(n) = x(n) + w(n) \Rightarrow$ DTS

3) Shift operator

"Integration" for continuous time system.

"Time-Shift" for Discrete time System.



* various configurations in which the System can be realised

i) Direct form - I ii) Direct-form-II

iii) Cascade form iv) Parallel form

System Realisation

Consider an LTI recursive DTS

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \Big/ 1 + \sum_{k=1}^{N} a_k z^{-k} \quad \text{---} (1)$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

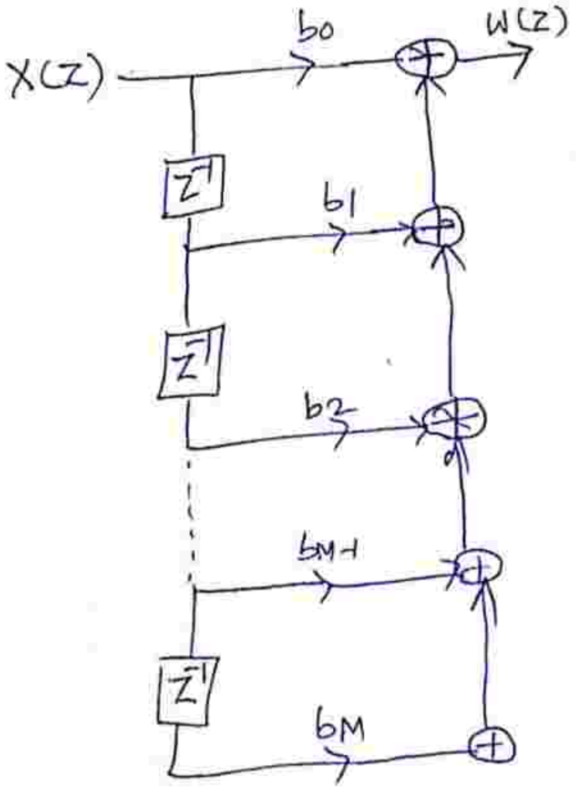
$$\Rightarrow Y(z) + Y(z) a_1 z^{-1} + Y(z) a_2 z^{-2} + \cdots + Y(z) a_N z^{-N} =$$

$$b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) \cdots + b_M z^{-M} X(z)$$

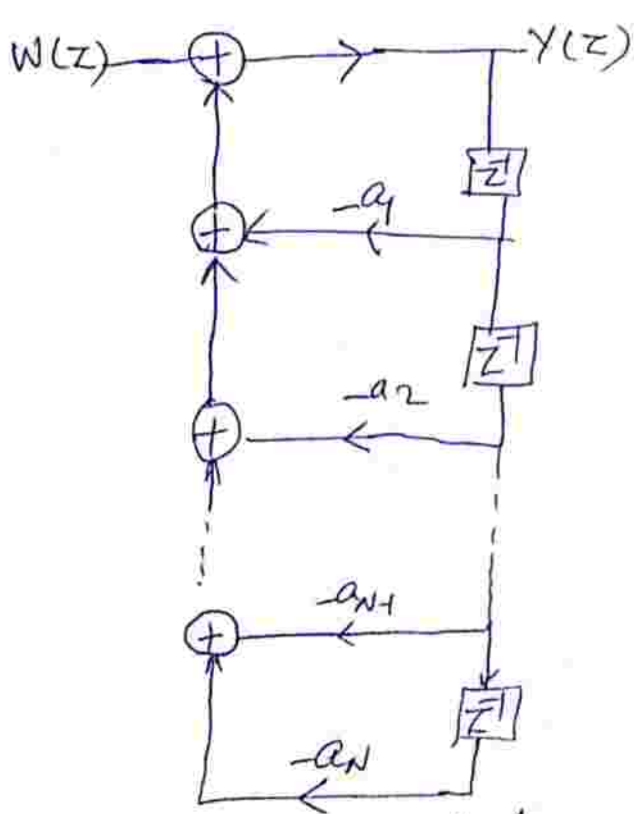
Let $b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_M z^{-M} X(z) = W(z)$

Then $Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + b_N z^{-N} Y(z) = W(z)$

$$Y(z) = W(z) - \left[a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) \cdots + b_N z^{-N} Y(z) \right]$$



i/p realisation



O/P realisation

→ combine the i/p and o/p realization, and get complete realization of syst using Direct-Form – I

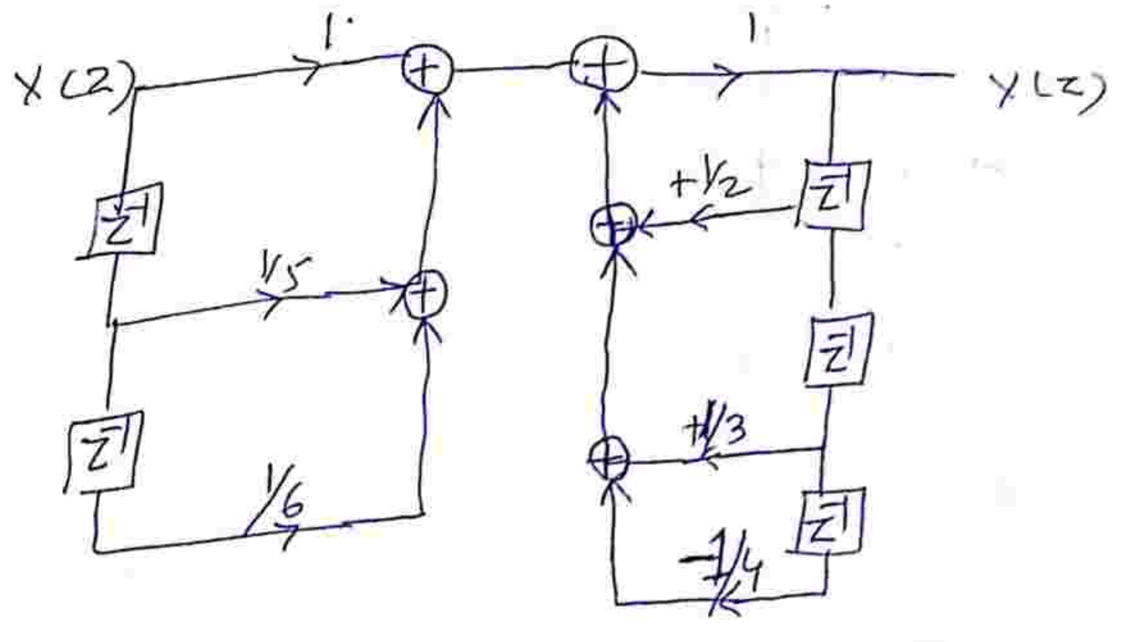
Ex Obtain the direct-form-I realization for the syst described by difference equation

$$y(n) - \frac{1}{2}y(n-1) - \frac{1}{3}y(n-2) + \frac{1}{4}y(n-3) = x(n) + \frac{1}{5}x(n-1) + \frac{1}{6}x(n-2)$$

Taking Z-transform both sides

$$Y(Z) - \frac{1}{2}Z^{-1}Y(Z) - \frac{1}{3}Z^{-2}Y(Z) + \frac{1}{4}Z^{-3}Y(Z) = X(Z) + \frac{1}{5}Z^{-1}X(Z) + \frac{1}{6}Z^{-2}X(Z)$$

$$\boxed{\frac{Y(Z)}{X(Z)} = \frac{1 - \frac{1}{2}Z^{-1} - \frac{1}{3}Z^{-2} + \frac{1}{4}Z^{-3}}{1 + \frac{1}{5}Z^{-1} + \frac{1}{6}Z^{-2}}}$$



System Realisation Using Direct-form-II

→ The major advantage of direct form-II Structure realisation is that the no. of delay is reduced by half. Hence, the system complexity drastically reduces the number of memory elements.

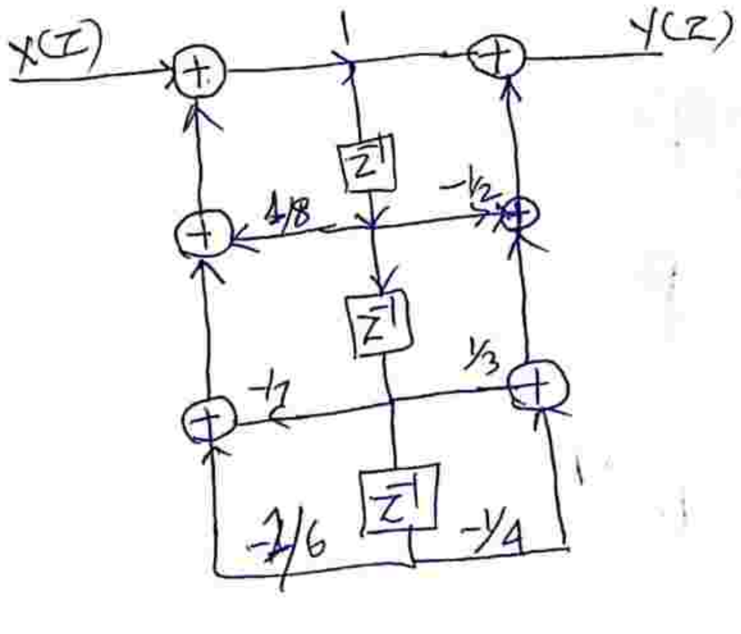
Ex Obtain the direct-form II realisation for the system described by the difference equation

$$y(n) - \frac{1}{8}y(n-1) + \frac{1}{7}y(n-2) + \frac{1}{6}y(n-3) = x(n) - \frac{1}{2}x(n-1) + \frac{1}{3}x(n-2) - \frac{1}{4}x(n-3)$$

Taking z-transform

$$Y(z) - \frac{1}{8}z^{-1}Y(z) + \frac{1}{7}z^{-2}Y(z) + \frac{1}{6}z^{-3}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z) + \frac{1}{3}z^{-2}X(z) - \frac{1}{4}z^{-3}X(z)$$

$$Y(z)\left[1 - \frac{1}{8}z^{-1} + \frac{1}{7}z^{-2} + \frac{1}{6}z^{-3}\right] = X(z)\left[1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} - \frac{1}{4}z^{-3}\right]$$



$$\frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2}z^{-1} \ldots}{1 - \frac{1}{8}z^{-1} \ldots}$$

Scanned by CamScanner

Cascade form

→ In cascade realisation, the system function is expressed as a product of several sub-system functions.

→ Each cascade sub-system in the cascade form is realised in direct-form II.

Ex Obtain the cascade realisation of the system described by the difference equations

$$y(n) + \frac{1}{16}y(n-1) + \frac{1}{6}y(n-2) - \frac{1}{24}y(n-3) - \frac{1}{16}y(n-4) = x(n) + \frac{5}{6}x(n-1) + x(n-2) + \frac{13}{36}x(n-3) + \frac{1}{6}x(n-4).$$

Taking z-transform

~~Y(z) +~~

$$Y(z)\left[1 + \frac{1}{16}z^{-1} + \frac{1}{6}z^{-2} - \frac{1}{24}z^{-3} - \frac{1}{16}z^{-4}\right] = X(z)\left[1 + \frac{5}{6}z^{-1} + z^{-2} + \frac{13}{36}z^{-3} + \frac{1}{6}z^{-4}\right]$$

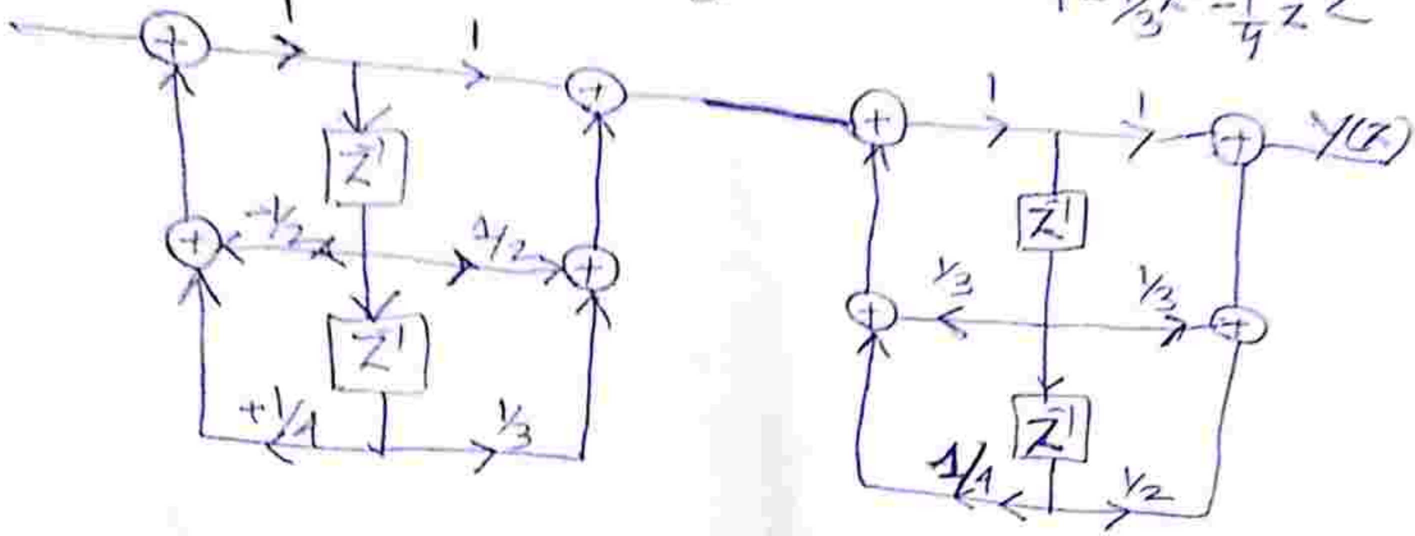
$$\boxed{\frac{Y(z)}{X(z)} = \frac{1 + \frac{5}{6}z^{-1} + z^{-2} + \frac{13}{36}z^{-3} + \frac{1}{6}z^{-4}}{1 + \frac{1}{16}z^{-1} + \frac{1}{6}z^{-2} - \frac{1}{24}z^{-3} - \frac{1}{16}z^{-4}}}$$

$$\frac{Y(z)}{X(z)} = \frac{\left(1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}\right)\left(1 + \frac{1}{3}z^{-1} + \frac{1}{2}z^{-2}\right)}{\underbrace{\left(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)}\underbrace{\left(1 + \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2}\right)}}$$

$$H(z) = H_1(z) \times H_2(z)$$

Pag

$$H_1(z) = \left(\frac{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}}{1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}} \right) \qquad H_2(z) = \frac{1 + \frac{1}{3}z^{-1} + \frac{1}{2}z^{-2}}{1 - \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2}}$$



* For recursive realisation, the current o/p $y(n)$ is a function of past o/ps, past and present inputs. This corresponds to IIR (Infinite Impulse Response)
* For non-recursive realisation, current o/p sample $y(n)$ is a function of only past and present inputs. This corresponds to FIR (Finite Impulse Response) digital filter.

⇒ For <u>direct form - I</u>

- Used separate delays for both input & output.
- This realisation requires M+N+1 multiplications, M+N additions and M+N+1 Memory locations.

⇒ for <u>direct form - II</u>

- It requires M+N+1 multiplications, M+N additions and max of {M, N} memory locations. Direct form - II realisation minimises the number of memory locations. It is said to be canonic.

Signal Flowgraph

→ A signal flowgraph is a graphical representation of the relationship between the variables of a set of linear difference equations.

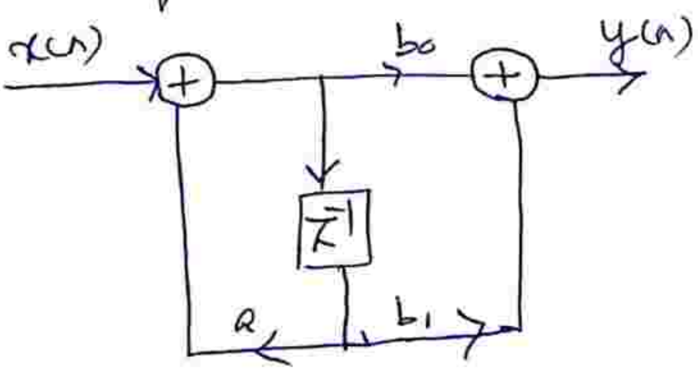
→ Basic elements are branches & nodes.

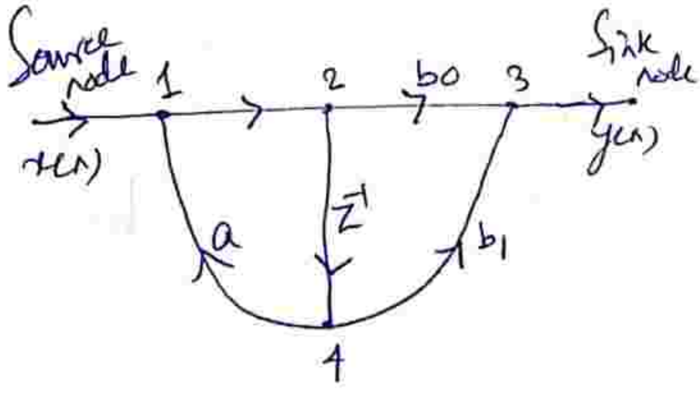
→ A node represents a syst. variable, which is equal to the sum of incoming signals from all branches connecting to that node.

→ Two types of nodes i.e Source node & sink node.

⇒ Source nodes are nodes that have no entering branches.
Sink nodes are nodes that have only entering branches.

→ The delay is indicated by the branch transmittance z^{-1}. When the branch transmittance is unity, it is left unbalanced.





Transposition theorem and transposed Structure

* Transpose of a Structure is defined by the following operations
  - i) Reverse the direction of all branches in the SFG.
  - ii) Interchange the i/ps & o/ps
  - iii) Reverse the roles of all nodes in the flowgraph.
  - iv) Summing points become branching points
  - v) Branching points become summing points.

** A realisation is canonic if the number of delay units used in the realisation is equal to the order of the transfer function realised. Canonic realisation has no redundant delay units.

Equivalent (Transposed) Structures

→ Two realisations are said to be equivalent if they have the same transfer function. A simple way to generate an equivalent Structure from a given realisation is via the transpose operation, which is as follows

→ Interchange the input & output nodes

→ Reverse all paths

→ Replace pick-off nodes by adders and vice-versa.

Consider the causal linear time-invariant (LTI) System with System function

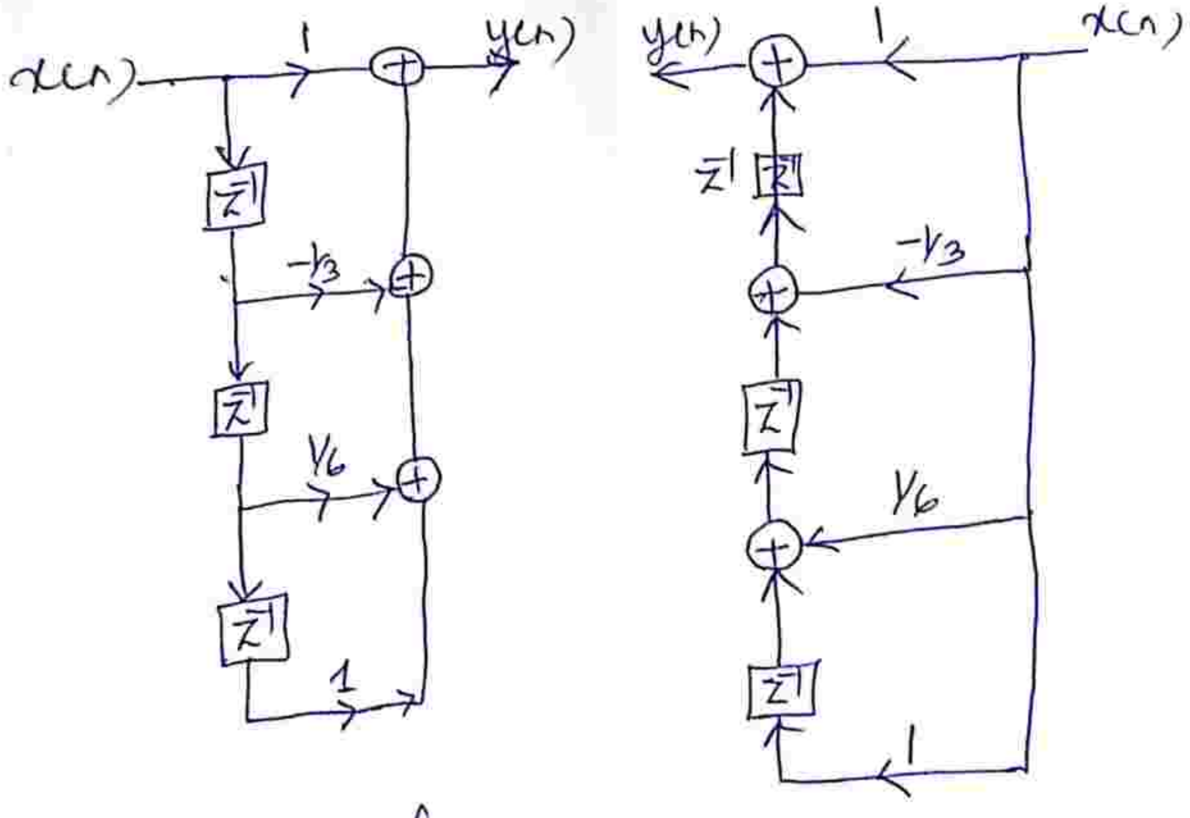
$$H(z) = 1 - \frac{1}{3}z^{-1} + \frac{1}{6}z^{-2} + z^{-3}$$

Draw the Direct form and transposed Direct form representation of this System.

Sol^n

$$H(z) = \frac{Y(z)}{X(z)} = 1 - \frac{1}{3}z^{-1} + \frac{1}{6}z^{-2} + z^{-3}$$

$$Y(z) = X(z) - \frac{1}{3}z^{-1}X(z) + \frac{1}{6}z^{-2}X(z) + z^{-3}X(z)$$



Direct-form

Transposed Version

Q.9 Calculate the percentage saving in calculating in a 512-point radix-2 FFT, when compared to direct DFT.

Sol: Direct computation of DFT.

No. of complex additions = $512 \times (511) = 261632$

No. of complex multiplications = $512^2 = 262144$

Radix-2 FFT.

No. of complex additions = 4600

No. of complex multiplications = 2304.

Percentage saving.

Percentage saving in additions = $100 - \frac{\text{No. of additions in r-2 FFT}}{\text{No. of add}^n \text{ in direct DFT}}$

$= 100 - \frac{4600}{261632} \times 100 = 98.2\%$

Percentage saving in multiplications = $100 - \frac{\text{No. of additions in r-2}}{\text{No. of add}^n \text{ in di. DFT}}$

$= 99.\ \%$

Q.10 Compare the DIT and DIF radix-2 FFT. (Ans)

DIT radix-2 FFT	DIF radix-2 FFT
1. The time domain sequence is decimated	1. The frequency domain sequence is decimated.
2. The input should be in bit reversed order and the output should be in normal order.	2. The input should be in normal order and the output should be in bit reversed order.
3. In each stage of computations, the phase factors are multiplied before add and subtract operation.	3. In each stage of computations the phase factors are multiplied after add and substract operations.
4. The value of N should be expressed such that $N=2^m$ and this algorithm consists of m stages of computations.	4. The value of N should be expressed such that $N=2^m$ and this algorithm consists of m stages of computations.
5. $N \log_2 N$ complex additions and $\frac{N}{2} \log_2 N$ complex multiplications are required.	5. $N \log_2 N$ complex additions and $\frac{N}{2} \log_2 N$ complex multiplications are required.

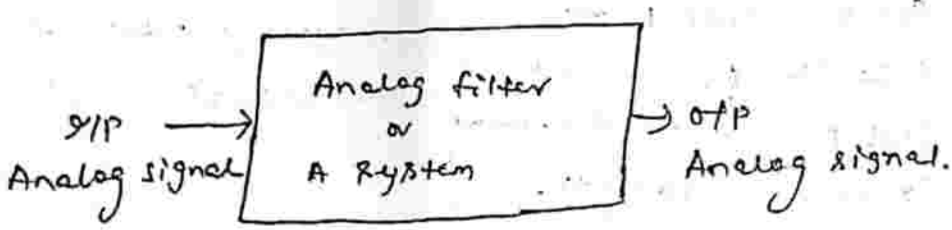
Design of Digital Filters.

A filter is essentially a network that selectively changes the waveshape of a signal in a desired manner. The objective of filtering is to improve the quality of a signal (for example, to remove noise) or to extract information from signals.

Filters are of 2 types → Analog filters
→ Digital filters.

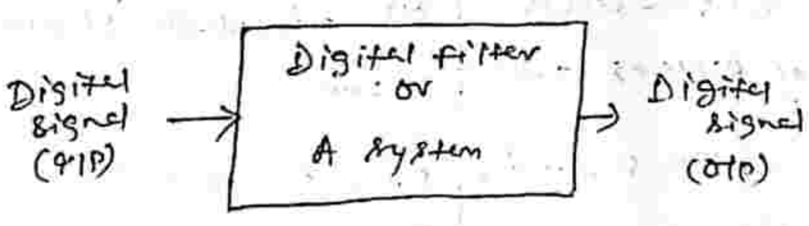
Analog filter:-

Analog filter may be defined as a system in which both the input and the output are continous time signals. (Analog signals)



Digital filters:-

Digital filter may be defined as a system in which both i/p and o/p's are digital signals.



Types of Analog filters:-

(i) Low pass (LPF)
(ii) High pass (HPF)
(iii) Band pass (BPF)
(iv) Band stop (BSF)
(v) All pass filters.

Types of Digital filters

(i) FIR
(ii) IIR

34 Q.1 Difference betn Analog and Digital filter:-

Analog filter	Digital filter
(i) A system in which both input and output signals are Analog.	(i) A system in which both input and output signals are digital.
(ii) Implementation of such filters is carried out by using active and passive components. Like Resistor(R), Capacitor(C), inductor (L) and active components like OPAMP, BJT etc.	(ii) Digital filters are implemented on a Computer or on a digital system. There are 3 basic elements for implementing a digital filter such as Adder, Multiplier, Delay elements.
(iii) Theoritically analog filters operate in infinite frequency range but practically it is limited to some megahertz. For example:- OPAMP frequency response is limited to 10MHz.	(iii) In case of digital filters frequency range is restricted to half of the sampling rat
(iv) Main disadvantages of Analog filters are A. Higher noise sensitivity B. Lack of flexibility C. Errors generated due to active and passive components.	(iv) Advantages of digital filter are A. Insensitive to noise B. flexibility in software design. C. High accuracy d. reliability
(v) There are 5 types of AF LPF, BPF, BSF, HPF and allpass filters	(v) There are 2 types FIR and IIR

Basics of Digital filter:-

A digital filter is a mathematical algorithm implemented in hardware/software. we can easily change the characteristics of the filter according to requirement just by changing the algorithm. Digital filters play very important role in DSP. Compared with analog filters they are preferred in a number of applications like data compression, speech processing, image processing etc. Because of the following advantages.

Advantages:- (over Analog filters)

(i) Digital filters have linear phase characteristics

(ii) performance of DF's does not vary with environmental changes for example thermal variations.

(iii) The frequency response of a digital filter can be adjusted if it is implemented using a programmable processor.

(iv). Several input signals can be filtered by one digital filter without the need to replicate the hardware.

(v) Digital filters can be used at very low frequencies

Disadvantages (over Analog)

(i) speed limitation

In case of digital filters, ADC and DAC are used. So the speed of digital filter depends on the conversion time of ADC and the settling time of DAC. Also depends upon the speed of processor.

(ii) Finite word length

The accuracy of digital filter depends on the word length used to encode them in binary form. word length

Should be long enough to obtain required accur.

(iii) Long design and development time.

An initial design and development time for digit. hardware is more than analog filter.

Importan concept on digital filters (must read)

A discrete time filter produces a discrete time output sequence y(n) for the discrete time input sequence x(n). A filter may be required to have a given frequency response or a specific response to an impulse, step or ramp, or stimulate analog system.

Depending on the form of the unit pulse response of the system digital filters are classified as

(i) Finite duration unit pulse response (FIR) filters.

Here the impulse response is of finite duration i.e it has a finite number of non-zero terms.

$$h(n) = \begin{cases} 1, & |n| \le 4 \\ 0, & \text{otherwise} \end{cases}$$

here it has only finite no. of non-zero terms.

(ii) Infinite duration unit pulse response (IIR) filters.

Here the impulse response is of infinite duration i.e it has infinite no. of non-zero terms.

$h(n) = a^n u(n)$ for $n \geq 0$ it has infinite no. of non-zero terms.

IIR filters are generally implemented

using structures having feed back (recursive structures - poles and zeros) and FIR filters are usually implemented using structures with no feed back (non-recursive structure - all zeros).

The response of a FIR filter depends only on present and past input samples, where as for the IIR filter, the present response is a function of present input, past input and past outputs.

Q.2 Difference between FIR and IIR filters.

FIR Filters	IIR Filters
(i) Filters having finite duration impulse response.	(i) Filters having infinite duration impulse response.
(ii) Ex. $h(n) = a^n \{ \text{for } n=0,1,2$	(ii) Ex. $h(n) = \{ a^n u(n)$
(iii) These filters have linear phase characteristics.	(iii) These filters have non linear phase characteristics.
(iv) No concept of feedback.	(iv) concept of feed back is there
(v) These are realized by using non-recursive method.	(v) These filters are realised by using recursive method.
(vi) These filters have a greater flexibility to control the shape of their magnitude.	(vi) These filters have less flexibility for obtaining non-standard frequency responses.
(vii) FIR filters cannot be derived from analog filters.	(vii) IIR filters are always derived from analog filters.

Causality and its implications :-

Impulse Response of Ideal Low pass filter :-

Here our objective is to see why practically Ideal filters are not realizeable.

Two important characterstics of Ideal filters:

(i) Ideal filters have constant gain in the passband and zero in the stop band.

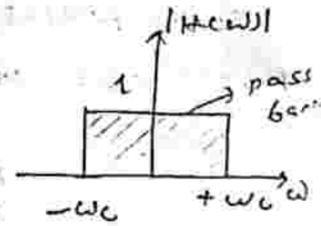
(ii) Ideal filter has linear phase response.

Please note that

In order to realize (design) digital filter, an important condition is that; the response filter should be causal.

Let us consider the issue of causality in more detail by examining the impulse response $h(n)$ of an ideal low pass filter with frequency response characteristic

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$



Now the value of $h(n)$ is obtained by taking inverse fourier of $H(\omega)$. (Magnitude Response of Ideal LPF)

According to the definition of inverse fourier we have

$$h(n) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} d\omega \right\}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 . e^{j\omega n} d\omega$$

Condition 1- for $n = 0$

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

$$\boxed{h(n) = \frac{\omega_c}{\pi}}, \; n = 0$$

Scanned by CamScanner

Condition II. $n \neq 0$

$$h(n) = \frac{1}{2\pi}\int_{-\omega c}^{\omega c} e^{j\omega n} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega c}^{\omega c} = \frac{1}{2\pi jn}\left[e^{j\omega cn} - e^{-j\omega cn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega cn} - e^{-j\omega cn}}{2j}\right]$$

But according to Euler's Identity

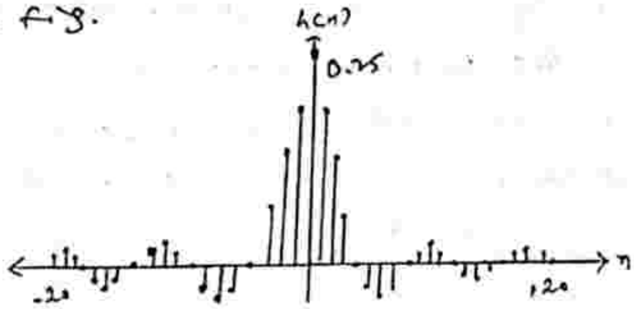
$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta \quad \text{For this.} \quad \frac{e^{j\omega cn} - e^{-j\omega cn}}{2j} = \sin \omega cn$$

$$h(n) = \frac{\sin \omega cn}{\pi n} \quad \text{.. when } n \neq 0.$$

Thus combing these two conditions

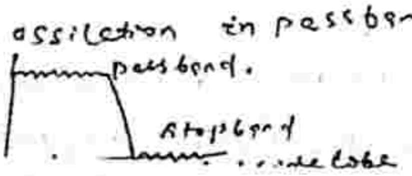
$$h(n) = \begin{cases} \frac{\sin \omega cn}{\pi n} & \text{for } n \neq 0 \\ \frac{\omega c}{\pi} & \text{for } n = 0 \end{cases}$$

Taking diffent valueof n and $\omega c = \frac{\pi}{4}$, If we plot the fig.



response h(n) of Ideal LPF.

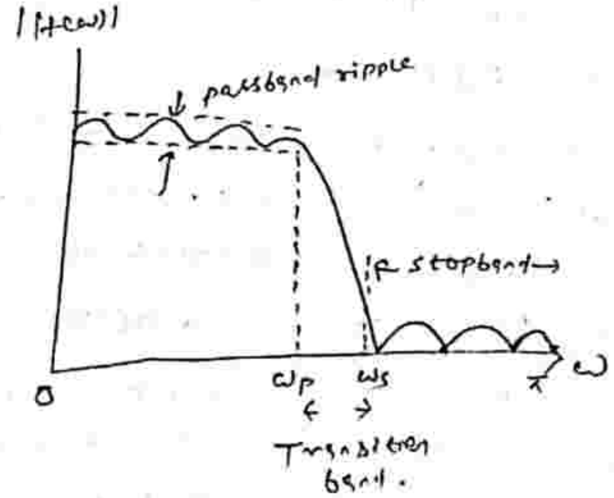
It is clear that ideal low pass filter is non causal and hence it cannot be realised in practice. one possible solution is to introduce a large delay no in h(n) and arbitrarily set h(n)=0, For $n \leq$ no. However the resulting system no longer has an ideal frequency response. If we set h(n)=0 for n < no the fourier series expansion H(w) results in the Gibb's phenomenon. i.e it will produce an ossilation in passband as well as in the stop band.



Although this discussion is limited to low pass filter we can conclude it in general for all other ideal filter characteristics. So the Ideal filter characteristics are non-causal and physically unrealizeable.

Characteristics of practical frequency selective filters:-

- As we observed from our discussion of the preceding section, ideal filters are non causal and hence physically unrealizeable for real-time signal processing applications.
- Causality implies that the frequency response characteristics $H(\omega)$ of the filter cannot be zero, except at a finite set of points in the frequency range.
- In addition, $H(\omega)$ canot have an infinitely sharp cutoff from passband to stopband, that is $H(\omega)$ cannot drop from from unity to zero abruptly.
- Some times the ideal filter frequency response though desireable but not necessary for practice applications. So we relax some conditions i.e the has some ripple in the passband as well as in the stop band which is tolerable.



- The transion of the frequency response from passband to stop band defines the transition band or transition region of the filter.
- ω_p defines the bandedge of passband and ω_s defines the starting of stopband so $\omega_s - \omega_p$ is the transit

4 copy

IIR Filter design by Bilinear Transformation Method. 147

IIR filter design using impulse invariant method are appropriate for the design of low pass and band pass filters whose resonant frequencies are low. This technique is not suitable for high pass and band reject filters. This limitation is overcome in the mapping technique called bilinear transformation. This transformation is one-to-one mapping from s-domain to the z-domain.

The bilinear transformation is obtained by using trapezoidal rule for numerical integration.

Let the system function of the analog filter be

$$H(s) = \frac{b}{s+a} \quad \text{——} \ (1)$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$\Rightarrow Y(s)(s+a) = bX(s)$$

$$\Rightarrow sY(s) + aY(s) = bX(s)$$

Taking inverse Laplace transform.

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Integrating both sides with in the limit $(nT-T)$ and nT

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} ay(t) = \int_{nT-T}^{nT} bx(t)\,dt \quad \text{—} (2)$$

The trapezoidal for numerical integration is given by

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}\left[a(nT) + a(nT-T)\right] \quad \text{—} (3)$$

Applying eqn (3) in (2)

$$\Rightarrow \ldots \frac{aT}{2} y(nT) + \frac{aT}{2} y(nT-T) = \frac{bT}{2}x(nT) + \frac{bT}{2}x(n\ldots$$

Taking Z-Transform, the system function of the digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \quad —④$$

Comparing eqⁿ ① and ④

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Bilinear Transformation

$$H(z) = H(s)\Big|_{at\ s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

put $z = re^{j\omega}$

$$s = \frac{2}{T}\left(\frac{1-r^{-1}e^{-j\omega}}{1+r^{-1}e^{-j\omega}}\right)$$

$$= \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

$$= \frac{2}{T}\left(\frac{r\cos\omega + rj\sin\omega - 1}{r\cos\omega + rj\sin\omega + 1}\right)$$

$$= \frac{2}{T}\left(\frac{[(r\cos\omega - 1) + jr\sin\omega][(r\cos\omega + 1) - jr\sin\omega]}{[(r\cos\omega + 1) + j(r\sin\omega)][(r\cos\omega + 1) - j(r\sin\omega)]}\right)$$

$$= \frac{2}{T}\left[\frac{(r\cos\omega - 1)(r\cos\omega + 1) - jr\sin\omega(r\cos\omega - 1) + jr\sin\omega(r\cos\omega + 1) + j^2 r^2\sin^2\omega}{(r\cos\omega + 1)^2 + r^2\sin^2\omega}\right]$$

$s = \sigma + j\Omega$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega + r\cos\omega - r\cos\omega - 1 - jr^2\cos\omega\sin\omega + jr\sin\omega + jr^2\sin\omega\cos\omega + jr\sin\omega + j^2 r^2\sin^2\omega}{(r\cos\omega + 1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2 - 1 + j2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right]$$

$$s = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

since $s = \sigma + j\Omega$

Therefore

$$\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right) \quad \text{and} \quad \Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) \quad ⑤$$

From eqn ⑤a it can be noted that if $r<1$, then $\sigma<0$, and if $r>1$, then $\sigma>0$. Thus, the left-half of the s-plane maps onto the points inside the unit circle in the z-plane and transformation results in a stable digital system. Consider eqn ⑤b for unit magnitude $(r=1)$, σ is zero. In this case,

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin(\omega/2)\cdot\cos(\omega/2)}{\cos^2(\omega/2)+\sin^2(\omega/2)+\cos^2(\omega/2)-\sin^2(\omega/2)}\right)$$

$$\Omega = \frac{2}{T}\tan\omega/2$$

or equivalently $\omega = 2\tan^{-1}\frac{\Omega T}{2}$

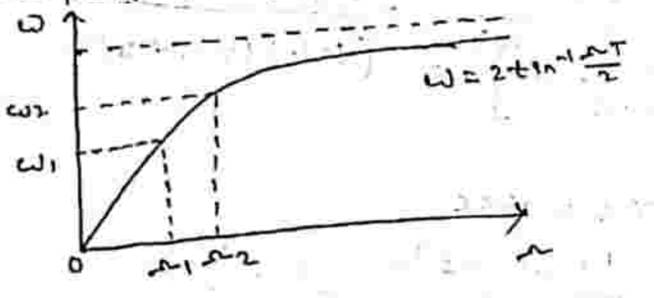
where Ω is called the analog frequency and ω is called the digital frequency.

i.e The transformation of analog frequency domain to digital frequency domain to digital frequency domain

$$\boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)}$$

The above eqn gives the relationship between the frequencies in the z domain.

If we plot the graph between Ω and ω.



it can be noted that the entire range of Ω is map only once into the range $-\pi<\omega<\pi$. However the mapping is non-linear and the lower frequencies in the analog domain are expanded in the digital domain, where as the higher frequencies are compressed. And this is called frequency warping.

50 Q.6 Convert the analog filter with system function

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9}$$ into a digital IIR

filter using bilinear transformation. The digital filter should have a resonant frequency of $\omega_r = \frac{\pi}{4}$

Sol:

Here the system function $H(s) = \frac{s+0.1}{(s+0.1)^2+(3)^2}$,

So $\Omega_c = 3$. The samping period T can be determined by using the formulae.

$$\Omega_r = \frac{2}{T} \tan\frac{\omega_r}{2}$$

The samping period is obtained from the above equation using.

$$T = \frac{2}{\Omega_c} \tan\frac{\omega_r}{2} = \frac{2}{3} \tan\frac{\pi}{8} = 0.276s$$

Using bilinear transformation

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{\frac{2}{T}\frac{(z-1)}{(z+1)} + 0.1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)} + 0.1\right]^2 + 9} = \frac{(2/T)(z-1)(z+1) + (z+1)^2 0.1}{[(2/T)(z-1) + 0.1(z+1)]^2 + 9(z+1)^2}$$

substituting $T = 0.276s$,

$$H(z) = \frac{1 + 0.027z^{-1} - 0.973z^{-2}}{8.572 - 11.84z^{-1} + 8.177z^{-2}}$$ (Ans)

Q.7 Apply bilinear transformation to

$$H(s) = \frac{2}{(s+1)(s+3)}$$ with $T = 0.1s$.

Sol:- for bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2(z-1)}{T(z+1)}}.$$

$$= \frac{2}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+1\right]\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+3\right]}$$

Using $T=0.1s$ $\quad H(z) = \dfrac{2}{\left(20\frac{(z-1)}{(z+1)}+1\right)\left(20\frac{(z-1)}{(z+1)}+3\right)}$

$$= \frac{2(z+1)^2}{(21z-19)(23z-17)}$$

Simplifying further

$$H(z) = \frac{0.0041(1+z^{-1})^2}{1-1.644z^{-1}+0.668z^{-2}}$$

Q.8 A digital filter with a 3dB bandwidth of 0.25π is to be designed from the analog filter whose system response is

$$H(s) = \frac{\Omega_c}{s+\Omega_c}$$ use bilinear transformation and

obtain $H(z)$.

Solⁿ:- we know $\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2} = \frac{2}{T}\tan 0.125\pi = 0.828/T$

The system response of the digital filter is given by

$$H(z) = H(s)\Big|_{s=\frac{2(z-1)}{T(z+1)}}$$

$$= \frac{\Omega_c}{\frac{2}{T}\frac{(z-1)}{(z+1)}+\Omega_c} = \frac{\frac{0.828}{T}}{\frac{2(z-1)}{T(z+1)}+\frac{0.828}{T}} = \frac{0.828(z+1)}{2(z-1)+0.828(z+1)}$$

Simplifying we get further $H(z) = \dfrac{1+z^{-1}}{3.414-1.414z^{-1}}$ (Ans)

Q.9 Using bilinear transformation obtain $H(z)$ if

$$H(s) = \frac{1}{(s+1)^2}$$ and $T=0.1s$.

Solⁿ For the bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+1\right]^2}$$ substituting $T=0.1s$,

R. Ranch.

$$H(z) = \frac{1}{\left[20\frac{(z-1)}{(z+1)}+1\right]^2} = \frac{(z+1)^2}{(21z-19)^2}$$

Futher simplyfying, $H(z) = \dfrac{0.0476(1+z^{-1})^2}{(1-0.9048z^{-1})^2}$ (Ans)

Design of Linear phase FIR filters:-

FIR stands for finite impulse response. FIR filters are called as non-recursive filters because the do not use feedback. Before studying the design of FIR filters; we will discuss some important characteristics of FIR filters.

FIR filters are inherently stable:-

We know that a system is said to be stable if bounded input produces bounded output (BIBO)

we have the difference equation of FIR filter,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \quad -①$$

Taking Z-Transform of eqn(1) we get

$$Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z) \quad -\cancel{(2)}$$

Now transfer function $H(z) = \dfrac{Y(z)}{X(z)}$

Thus from eqn (2) we get

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M-1} b_k z^{-k}$$

Taking inverse Z-Transform $h(n) = \begin{cases} b_n & \text{for } 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \quad -②$

Expanding eqn(1), we get

$$y(n) = b_0 x(n) + b_1 x(n-1) + \dots + b_{M-1} x(n-M+1)$$

Using eqn(2) we get

$$y(n) = h_0 x(n) + h(1) x(n-1) + \dots + h(M-1) x(n-M+1) \quad -③$$

Here $h(0), h(1) \dots$ are constants that means they are bounded. Now from eqn(3) the output will be bounded if we apply bounded input. That means for every bounded input; output of FIR filter is bounded. Thus FIR filters are

Magnitude and phase response of Digital filters:

The discrete-time fourier transform of a finite sequence impulse response h(n) is given by

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n} = |H(e^{j\omega})| e^{j\phi(\omega)}$$

where $|H(e^{j\omega})|$ is called magnitude response and denoted as $M(\omega)$.

$$M(\omega) = \sqrt{Re[H(e^{j\omega})]^2 + Im[H(e^{j\omega})]^2}$$

and $\phi(\omega)$ is called phase response

and $\phi(\omega) = \tan^{-1}\left[\frac{Im(H(e^{j\omega}))}{Re[H(e^{j\omega})]}\right]$

Filter can have a linear or non-linear phase depending upon the delay functions, namely the phase delay and group delay. The phase and group delays of the filter are given by

$$\tau_p = -\frac{\phi(\omega)}{\omega} \text{ and } \tau_g = -\frac{d\phi(\omega)}{d\omega} \text{ respectively.}$$

Linear phase filters are those filters in which the phase delay and group delay are constants. i.e Independent of frequency. LPF are also called constant time delay filters.

Let us obtain the conditions FIR filters must satisfy in order to have constant phase and group delays and hence obtain the conditions for having a linear phase.

For phase response to be linear

$$\frac{\phi(\omega)}{\omega} = -\tau, \quad -\pi \le \omega \le \pi$$

Therefore $\phi(\omega) = -\omega\tau$

Where τ is the constant phase delay expressed in numbers of samples.

54 of samples.

$$\phi(\omega) = \tan^{-1}\frac{Im\, H(e^{j\omega})}{Re\, H(e^{j\omega})} = -\omega\tau \quad -①$$

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n}.$$

$$= \sum_{n=0}^{M-1} h(n)(\cos\omega n - j\sin\omega n)$$

$$= \underbrace{\sum_{n=0}^{M-1} h(n)\cos\omega n}_{Re\, H(e^{j\omega})} - j\underbrace{\sum_{n=0}^{M-1} h(n)\sin\omega n}_{Im\, H(e^{j\omega})}$$

In eqn ①

$$\frac{-\sum_{n=0}^{M-1} h(n)\sin\omega n}{\sum_{n=0}^{M-1} h(n)\cos\omega n} = \tan(-\omega\tau)$$

$$\Rightarrow \tan\omega\tau = \frac{\sum_{n=0}^{M-1} h(n)\sin\omega n}{\sum_{n=0}^{M-1} h(n)\cos\omega n}$$

$$\Rightarrow \frac{\sin\omega\tau}{\cos\omega\tau} = \frac{\sum_{n=0}^{M-1} h(n)\sin\omega n}{\sum_{n=0}^{M-1} h(n)\cos\omega n}$$

$$\Rightarrow \sum_{n=0}^{M-1} h(n)\cos\omega n\sin\omega\tau = \sum_{n=0}^{M-1} h(n)\sin\omega n\cos\omega\tau$$

i.e $\sum_{n=0}^{M-1} h(n)[\sin\omega\tau\cos\omega n - \cos\omega\tau\sin\omega n] = 0$

Therefore $\sum_{n=0}^{M-1} h(n)\sin(\omega\tau - \omega n) = 0 \quad -② \quad -1(A)$

A soln to eq ② gives $\tau = \frac{(M-1)}{2}$ and $-③$

$h(n) = h(M-1-n)$ for $0 < n < M-1$. $-④$

If eqn ③ and ④ are satisfied, then the FIR Filter will have constant phase and group delays and thus the phase of the filter will be linear. The phase and group delays of the linear phase FIR filter are equal and constant over the frequency band.

- Whenever a constant group delay alone is preferred, the impulse response will be of the form

$h(n) = -h(M-1-n)$ and is antisymmetric about the centre of the impulse response sequence.

- The applications of FIR filters like the wideband differentiator and Hilbert transformer use such antisymmetric impulse response sequences.

- Thus the condition for FIR filters to be linear phase for symmetric filters $h(n) = h(M-1-n)$ and for Antisymmetric filters $h(n) = -h(M-1-n)$.

Q.10 The Length of an FIR filter is 9. If the filter has a linear phase, show that $\sum_{n=0}^{M-1} h(n)\sin(\omega\tau - \omega n) = 0$ is satisfied.

Sol^n:- The length of the filter $M = 9$, Therefore,

$$\tau = \frac{M-1}{2} = 4.$$

For linear phase $h(n) = h(M-1-n)$. Therefore the filter coefficients are $h(0) = h(8)$, $h(1) = h(7)$, $h(2) = h(6)$, $h(3) = h(5)$ and $h(4)$

Now $$\sum_{n=0}^{M-1} h(n)\sin(\omega\tau - \omega n) = \sum_{n=0}^{8} h(n)\sin\omega(\tau - n)$$

$$= h(0)\sin 4\omega + h(1)\sin 3\omega + h(2)\sin 2\omega + h(3)\sin\omega + h(4)\sin 0$$
$$+ h(5)\sin(-\omega) + h(6)\sin(-2\omega) + h(7)\sin(-3\omega) + h(8)\sin(-4\omega)$$

Q.11 The following transfer function characterises FIR filter (M=11). Determine the magnitude response, show that the phase and group delays are constant.

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

Soln:- The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(10)z^{-10}$$

The phase delay, $\tau = \frac{M-1}{2} = 5$. since $\tau = 5$, the transfer function can be expressed as

$$H(z) = z^{-5}\left[h(0)z^{5} + h(1)z^{4} + h(2)z^{3} + \dots + h(9)z^{-4} + h(10)z^{-5}\right]$$

since $h(n) = h(M-1-n)$,

$$H(z) = z^{-5}\left[h(0)(z^{5}+z^{-5}) + h(1)(z^{4}+z^{-4}) + h(2)(z^{3}+z^{-3}) + h(3)(z^{2}+z^{-2}) + h(4)(z+z^{-1}) + h(5)\right]$$

The frequency response is obtained by replacing z with $e^{j\omega}$

$$H(e^{j\omega}) = e^{-j5\omega}\{h(0)[e^{5j\omega} + e^{-5j\omega}] + h(1)[e^{4j\omega} + e^{-4j\omega}] + h(2)[e^{3j\omega} + e^{-3j\omega}] + h(3)[e^{2j\omega} + e^{-2j\omega}] + h(4)[e^{j\omega} + e^{-j\omega}] + h(5)\}$$

$$= e^{-j5\omega}\left[h(5) + 2\sum_{n=0}^{4} h(n)\cos(5-n)\omega\right] = e^{-j5\omega} M(\omega)$$

so $M(\omega)$ is magnitude response and $e^{-j5\omega}$ is the phase response. $\phi(\omega) = -5\omega$.

Phase delay $\tau_p = -\frac{\phi(\omega)}{\omega} = 5$ and

Group delay $\tau_g = -\frac{d[\phi(\omega)]}{d\omega} = -\frac{d[-5\omega]}{d\omega} = 5$

which shows that the group delay and phase delay are constants.

band and width of the passband is called the bandwidth of the filter.

- In any filter design problem we can specify:
  (i) The maximum tolerable passband ripple.
  (ii) The maximum tolerable stopband ripple.
  (iii) The passband edge frequency ω_p
  (iv) Stopband edge frequency ω_s.

Based on these parameters and the value of (M,N) {a_k} and {b_k} are determined to get required frequency response $H(\omega)$.

where $$H(\omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\omega k}}{1+\sum_{k=1}^{N} a_k e^{-j\omega k}}$$

Design of IIR filters:-

In order to design the digital IIR filter, analog IIR filter is designed first. Then analog filter is converted into the digital filter. Here you may ask a question, why to design digital filter from analog filter?

The reasons are as follows:

1) The procedure to design analog filter is readily available and it is highly advanced.

2) when we design digital filter using analog filter then the implementation becomes simple.

Different methods used to design IIR filters:-

1. Approximation of derivatives
2. Impulse invariance method
3. Bilinear transformation
4. Matched Z-transform.
5. Least square filter design.

we will discuss method 2 and 3 as it is in syllabus.

IIR Filter design by Impulse Invariant Method.

The response of analog filter is always given by Laplace transform and response of an digital filt. is always given by Z-Transform.

Let the analog filter frequency response is given by $H(s)$

$$H(s) = \sum_{i=1}^{N} \frac{C_k}{s - P_k} \quad \text{--- (1)}$$

where $\{P_k\}$ are the poles of analog filter and $\{C_k\}$ are the co-efficients in the partial fraction expansion consequently,

$$h_a(t) = \sum_{k=1}^{N} C_k e^{P_k t}, \quad t \geq 0$$

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} C_k e^{P_k nT}$$

Now the system function

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} C_k e^{P_k nT} \right) z^{-n} = \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} (e^{P_k T} z^{-1})^n \quad \text{--- (2)}$$

The inner sum in the above eq? converges because P_k and yields.

$$\sum_{n=0}^{\infty} (e^{P_k T} z^{-1})^n = \frac{1}{1 - e^{P_k T} z^{-1}} \quad \text{--- (3)}$$

putting eq (3) in (2)

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}} \quad \text{--- (4)}$$

Now comparing eqn (1) and (4)

$$\frac{1}{s - P_k} \longleftrightarrow \frac{1}{1 - e^{P_k T} z^{-1}}$$

Some Standard Transformations are (i.e from S to z domain)

(i) $\dfrac{1}{s-p_k} \Rightarrow \dfrac{1}{1-e^{p_kT}z^{-1}}$

(ii) $\dfrac{1}{(s+s_i)^m} \rightarrow \dfrac{(-1)^{m-1}}{(m-1)!}\dfrac{d^{m-1}}{ds^{m-1}}\left[\dfrac{1}{1-e^{-sT}z^{-1}}\right]; \ s \rightarrow s_i$

(iii) $\dfrac{s+a}{(s+a)^2+b^2} \rightarrow \dfrac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$

(iv) $\dfrac{b}{(s+a)^2+b^2} \rightarrow \dfrac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$

Design steps for impulse invariance method:-

(i) H(s) given if not then find from specification.

(ii) If required expand H(s) by using partial fraction expansion (PFE)

(iii) obtain Z-transform of each PFE term using impulse invariance transformation equation.

(iv) Obtain H(z), this is required digital IIR filter.

Q.3 Convert the analog filter into a digital whose system function is

$$H(s) = \frac{s+0.2}{(s+0.2)^2+9}$$

use impulse invariant method. Assume T = 1s.

Sol^n:- The system response is ~~analogous to~~ the standard form $H(s) = \dfrac{s+a}{(s+a)^2+b^2}$

where $a=0.2$ and $b=3$. The system response of the digital filter can be obtained using the standard,

$$H(z) = \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$= \frac{1-e^{-0.2T}(\cos 3T)z^{-1}}{1-2e^{-0.2T}(\cos 3T)z^{-1}+e^{-0.4T}z^{-2}}.$$

Taking T=1s

$$H(z) = \frac{1-(0.8187)(-0.99)z^{-1}}{1-2(0.8187)(-0.99)z^{-1}+0.6703z^{-2}}$$

That is

$$H(z) = \frac{1+0.8105z^{-1}}{1+1.6210z^{-1}+0.6703z^{-2}}$$

Q.4 For the analog transfer function

$$H(s) = \frac{1}{(s+1)(s+2)}$$

determine H(z) using impulse invariant technique. Assume T=1s.

Sol^n: using partial fractions, H(s) can written as.

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = (s+1)\frac{1}{(s+1)(s+2)}\Big|_{s=-1} = 1$$

$$B = (s+2)\frac{1}{(s+1)(s+2)}\Big|_{s=-2} = -1$$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

The system function of the digital filter

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}}$$

$$= \frac{z^{-1}\left[e^{-T}-e^{-2T}\right]}{1-(e^{-T}+e^{-2T})z^{-1}+e^{-3T}z^{-2}}$$

Since T=1s $\quad H(z) = \dfrac{0.232z^{-1}}{1-0.503z^{-1}+0.1498z^{-2}}$

Q.5 Determine H(z) using impulse invariance method for system function,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Soln:- The Given transfer function is

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} \quad \text{---(i)}$$

In partial fraction H(s) can be written as

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{s+0.5} + \frac{Bs+C}{(s^2+0.5s+2)} \quad \text{---(ii)}$$

Let us obtain the values of A, B and C

$$\frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A(s^2+0.5s+2)+(Bs+C)(s+0.5)}{(s+0.5)(s^2+0.5s+2)}$$

$$\therefore A(s^2+0.5s+2)+(Bs+C)(s+0.5)=1$$

$$\therefore As^2+0.5sA+2A+Bs^2+B\,0.5s+C.s+0.5C=1$$

$$s^2(A+B)+s(0.5A+0.5B+C)+(2A+0.5C)=1 \quad \text{---(iii)}$$

Now s^2 term is absent in RHS

$$\therefore A+B=0 \quad \text{----(iv)}$$

Similarly 's' term is absent in R.H.S

$$\therefore 0.5A+0.5B+C=0 \quad \text{---(v)}$$

$$\text{and } 2A+0.5C=1 \quad \text{-----(vi)}$$

Now we will solve equations (iv), (v) and (vi) to obtain the values of A, B and C.

From eqn (iv), $B=-A$

Putting this value in equation (v) we get

$$0.5A-0.5A+C=0$$

$$\boxed{\therefore C=0}$$

From eqn (vi) $2A+0=1 \quad \therefore \boxed{A=0.5}$

© Soumya R. Pande.

Since $B = -A$ we get $\boxed{B = -0.5}$

Putting these values in eqn (ii) we get

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2}$$

To use the standard transformation formulaes we will convert the second term on R.H.S in the form

$$\frac{s+a}{(s+a)^2+b^2} \text{ and } \frac{b}{(s+a)^2+b^2}$$

Consider the term $s^2+0.5s+2$, it can be expressed as,

$$s^2+0.5s+2 = (s^2+0.5s+0.0625)+(1.9375)$$

$$\therefore s^2+0.5s+2 = (s+0.25)^2+(1.39)^2$$

Putting these values in equation (7) we get

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{(s+0.25)^2+(1.39)^2}$$

$$\therefore H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25-0.25}{(s+0.25)^2+(1.39)^2}\right]$$

$$\therefore H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(1.39)^2}\right] + 0.5\left[\frac{+0.25}{(s+0.25)^2+(1.39)^2}\right]$$

Now we want the numerator of third term equ to 1.39. it is arranged as follows,

$$H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(1.39)^2}\right] + \frac{0.5\times0.25}{1.39}\left[\frac{1.39}{(s+0.25)^2+(1.39)^2}\right]$$

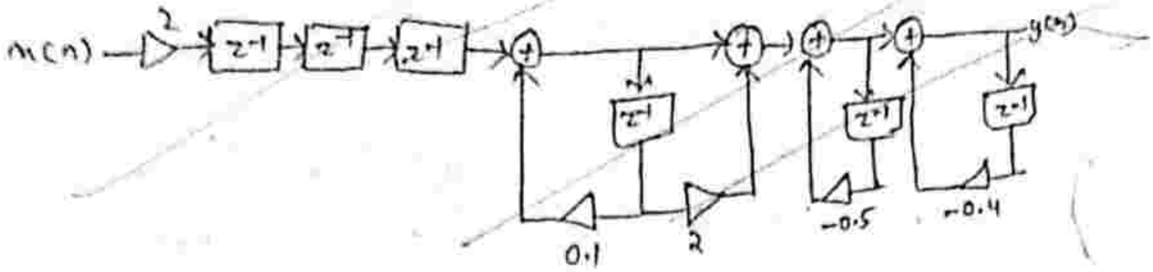
$$\therefore H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2+(1.39)^2}\right] + 0.089\left[\frac{1.39}{(s+0.25)^2+(1.39)^2}\right]$$

Now using the standard transformation formulae

$$H(z) = \frac{0.5}{1-e^{0.5T_s}\cdot z^{-1}} - 0.5\left[\frac{1-e^{-0.25T_s}[\cos 1.39T_s]z^{-1}}{1-2e^{-0.25T_s}[\cos 1.39T_s]z^{-1}+e^{-0.5T_s}z^{-2}}\right]$$

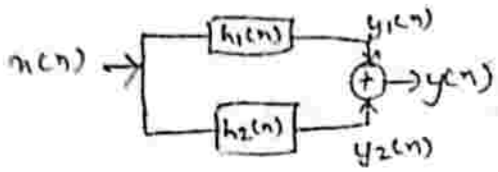
$$+0.089\left[\frac{e^{-0.25T_s}[\sin 1.39T_s]z^{-1}}{1-2e^{-0.25T_s}(\cos 1.39T_s)z^{-1}+e^{-0.5T_s}z^{-2}}\right]$$

The cascade realisation of system transfer function is



Parallel Realisation of IIR systems :-

If two systems are connected in parallel their individual transfer functions will be added.



$y_1(n) = x(n) * h_1(n)$ $y_2(n) = x(n) * h_2(n)$

$Y_1(z) = X(z) H_1(z)$ $Y_2(z) = X(z) H_2(z)$

$y(n) = y_1(n) + y_2(n)$

$\Rightarrow Y(z) = Y_1(z) + Y_2(z)$

$\Rightarrow Y(z) = [H_1(z) + H_2(z)] X(z)$

$\Rightarrow H(z) = \dfrac{Y(z)}{X(z)} = H_1(z) + H_2(z)$

The parallel realisation is useful for high speed filtering applications since the filter operation is performed in parallel, i.e. the processing is performed simultaneously.

Q.6 Determine the parallel realisation of the IIR digital filter transfer function

$$H(z) = \frac{3(2z^2+5z+4)}{(2z+1)(z+2)}$$

Ans:- In order to find the parallel realisation, the partial fraction expansion of $H(z)/z$ is first determined just we did for inverse Z-Transforms.

This gives

$$F(z) = \frac{H(z)}{z} = \frac{\frac{3}{2}(2z^2+5z+4)}{z(z+\frac{1}{2})(z+2)} = \frac{A_1}{z} + \frac{A_2}{z+\frac{1}{2}} + \frac{A_3}{z+2}$$

$$A_1 = zF(z)\Big|_{z=0} = \left.\frac{3/2(2z^2+5z+4)}{(z+\frac{1}{2})(z+2)}\right|_{z=0} = 6$$

$$A_2 = \left[z+\frac{1}{2}\right]F(z)\Big|_{z=-\frac{1}{2}} = \left.\frac{3/2(2z^2+5z+4)}{z(z+2)}\right|_{z=-\frac{1}{2}} = -4$$

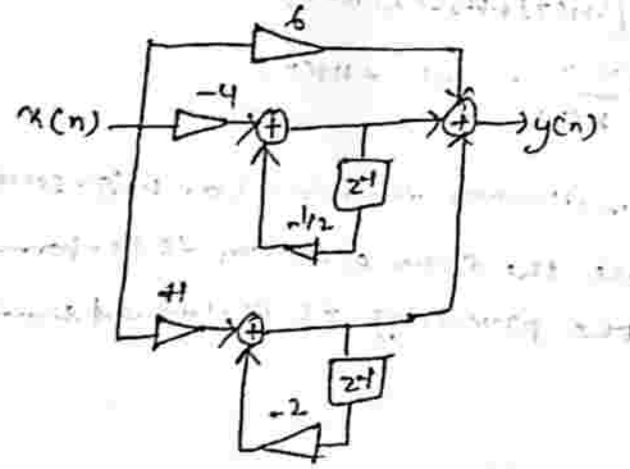
$$A_3 = (z+2)F(z)\Big|_{z=-2} = \left.\frac{\frac{3}{2}(2z^2+5z+4)}{z(z+\frac{1}{2})}\right|_{z=-2} = 1$$

Therefore $\frac{H(z)}{z} = \frac{6}{z} - \frac{4}{z+\frac{1}{2}} + \frac{1}{z+2}$

Hence $H(z) = 6 - \frac{4z}{z+\frac{1}{2}} + \frac{z}{z+2}$

$$= 6 - \frac{4}{1+\frac{1}{2}z^{-1}} + \frac{1}{1+2z^{-1}}$$

The parallel realisation is



Q.7. Determine the parallel realisation of the IIR digital filter transfer function

$$H(z) = \frac{3z(5z-2)}{(z+\frac{1}{2})(3z-1)}$$

Ans $H(z) = \frac{3z(5z-2)}{(z+\frac{1}{2})(3z-1)} = \frac{z(5z-2)}{(z+\frac{1}{2})(z-1/3)}$

$$\frac{H(z)}{z} = \frac{A_1}{z+1/2} + \frac{A_2}{z-\frac{1}{3}}$$

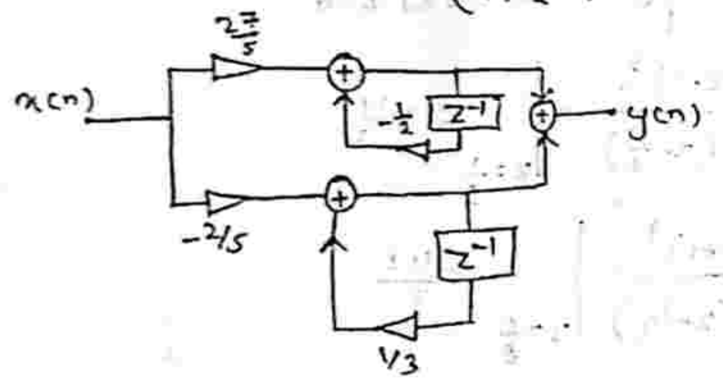
$$A_1 = \left(z+\frac{1}{2}\right)\left[\frac{z(5z-2)}{(z+\frac{1}{2})(z-1/3)}\right]_{z=-1/2}$$

$$= \frac{z(5z-2)}{(z-\frac{1}{3})} = \frac{27}{5}$$

$$A_2 = \left(z-\frac{1}{3}\right)\left[\frac{5z-2}{(z+\frac{1}{2})(z-\frac{1}{3})}\right] = -\frac{2}{5}$$

Therefore, $H(z) = \frac{27}{5}\frac{z}{(z+\frac{1}{2})} - \frac{2}{5}\frac{z}{(z-\frac{1}{3})}$

$$= \frac{27}{5}\frac{1}{(1+\frac{1}{2}z^{-1})} - \frac{2}{5}\frac{1}{(1-\frac{1}{3}z^{-1})}$$



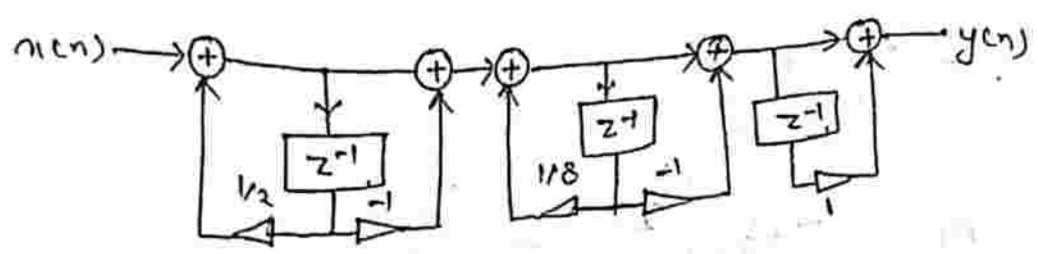
(Parallel realisation)

Q.8. Draw the structure of cascade and parallel realisations of

$$H(z) = \frac{(1-z^{-1})^3}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{8}z^{-1})}$$

Sol^n The given transfer function can be written in the form

$$H(z) = \frac{(1-z^{-1})}{(1-\frac{1}{2}z^{-1})} \cdot \frac{(1-z^{-1})}{(1-\frac{1}{8}z^{-1})} \cdot (1-z^{-1}) = H_1(z)H_2(z)H_3(z)$$



(Cascade realisation)

For finding the parallel realisation, the partial fraction expansion of $H(z)/z$ is determined as

$$F(z) = \frac{H(z)}{z} = \frac{(z-1)^3}{z^2(z-1/2)(z-\frac{1}{8})} = \frac{A_1}{z^2} + \frac{A_2}{z} + \frac{A_3}{z-1/2} + \frac{A_4}{z-\frac{1}{8}}$$

where

$$A_1 = \frac{(z-1)^3}{(z-1/2)(z-\frac{1}{8})}\Bigg|_{z=0} = -16$$

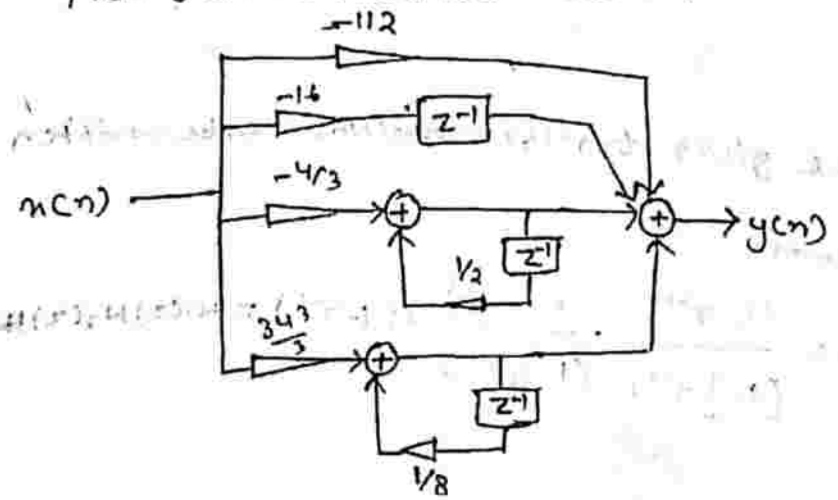
$$A_2 = \frac{d}{dz}\left[\frac{(z-1)^3}{(z-1/2)(z-\frac{1}{8})}\right]_{z=0} = -112$$

$$A_3 = \frac{(z-1)^3}{z^2(z-\frac{1}{8})}\Bigg|_{z=\frac{1}{2}} = -\frac{4}{3}$$

$$A_4 = \frac{(z-1)^3}{z^2(z-1/2)}\Bigg|_{z=\frac{1}{8}} = \frac{343}{3}$$

$$H(z) = -112 - 16z^{-1} - \frac{4}{3}\frac{1}{(1-\frac{1}{2}z^{-1})} + \frac{343}{3}\frac{1}{(1-\frac{1}{8}z^{-1})}$$

The function can be realised in parallel form as



Q.9 Draw the cascade and parallel realisation structures for the system described by the system function

$$H(z) = \frac{5\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

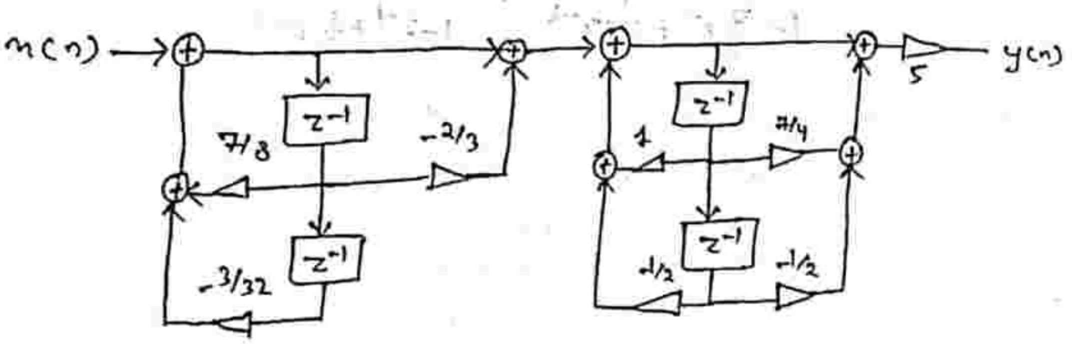
Solution:-

Cascade Realisation:- This realisation can be easily obtained by grouping poles and zeros of the system function in several possible ways; one possible pairing of poles and zeros is

$$H_1(z) = \frac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}}, \quad H_2(z) = \frac{1+\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

Hence $H(z) = 5H_1(z)H_2(z)$

Cascade Structure:



Parallel Realisation:- To obtain the parallel form $H(z)$ must be expanded in partial fractions. Thus we have

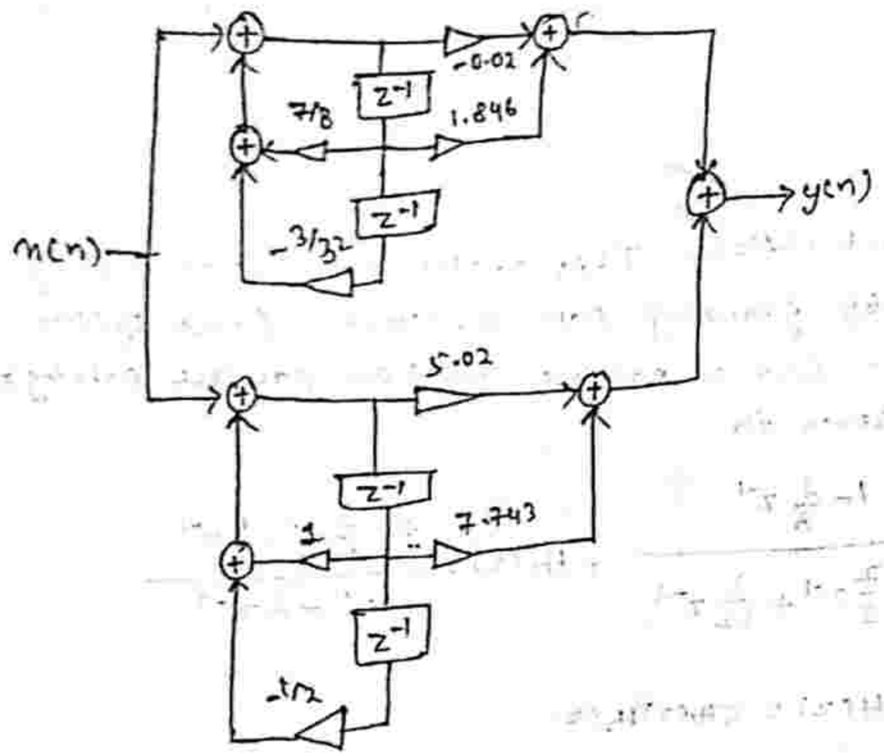
$$H(z) = \frac{A_1}{1-\frac{3}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{8}z^{-1}} + \frac{A_3}{1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}} + \frac{A_3^*}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

Upon solving, we find $A_1 = 2.933$, $A_2 = -2.947$

$A_3 = 2.507 - j10.45$ and $A_3^* = 2.507 + j10.45$

$$H(z) = \frac{2.933}{1-\frac{3}{4}z^{-1}} - \frac{2.947}{1-\frac{1}{8}z^{-1}} + \frac{2.507-j10.45}{1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}} + \frac{2.507+j10.45}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

Parallel Realisation:-



Above

$$H(z) = \frac{-0.02 + 1.846z^{-1}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}} + \frac{5.02 + 7.743z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Q.10. obtain the cascade and parallel realisation for the system function given by

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2})}$$

Solution:-

Cascade Realisation:- To obtain the cascade realisation, the transfer function is broken into a product of two functions as

$$H(z) = H_1(z) H_2(z)$$

where $H_1(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}}$ and $H_2(z) = \frac{1}{1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$